A NOTE ON ABSTRACT CONVEXITY AND ITS APPLICATIONS

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ABSTRACT

This paper presents a survey of some results from and applications of abstract convexity based on the notions of Minkowski Duality and sub differentials. In this paper, we also discuss different kinds of abstract convexities as order convexity, Simplicial Convexity , B′- Simplicial Convexity, mc- Space, c-Space and L- Space. We also discuss the relationship between these different abstract convexities.

Keywords: Simplicial Convexity , B′- Simplicial Convexity, Order Convexity, Minkowski Duality, Subdifferentials.

INTRODUCTION

Convexity has been increasingly important in recent years in the study of extremum problems in many areas of applied mathematics. The notion of convexity is a basic mathematical structure that is used to analyze many different problems. One of the main result of convex analysis asserts that an arbitrary lower semi-continuous function f is the upper envelope of the set of all its affine minorants:

$$f(x) = sup\{h(x): h is an affine function , h \leq f\}$$

The supremum above is attained iff the sub-differential of f at the point x is non-empty. Thus convexity can be defined as linearity with envelope representation. Many researchers have dealt with the problem of generalizing usual convexity from different points of view. This generalization can be categorized in two types. One which is motivated by concrete problems and the other which are stated from an axiomatic point of view. This second notion and the concept of envelope representation stimulated the development of rich theory of convexity without linearity, known as abstract convexity. In particular, functions which can be represented as upper envelopes of subsets of a set of sufficiently simple functions, are studied in this theory.

Many results from convex analysis which are related to various kinds of convex duality can be extended to Abstract Convex Analysis. Abstract Convexity sheds some new light to the classical Fenchel-Moreau duality and level sets conjugation.(Singer, 1997). Overview of the origin, basic abstract convexity and cone-vexing abstraction has been given. (Kukateladze, 2008). Also the characteristics of abstract convexity structures on topological spaces with selection property have been introduced.(Xiang & Xia, 2007). The notion of duality to arbitrary partially ordered sets and abstract sub differentials corresponding to such dualities has been considered.(Sharikov, 2009). The structure of both sub differential and abstract epsilon sub-differential has been discussed and it has been proved by the help of examples that some properties which hold for maximal monotone operators and their enlargements in the classical case also holds in abstract setting. (Regina & Rubinov, 2008). Some types of convergence of abstract convex functions have been studied and shown that under certain conditions, convergence of support sets of a sequence of abstract convex functions implies graph convergence of the support sets at a point and the global sub differential. (Loffe & Rubinov, 2002). New inequalities have been derived by sharpening well known inequalities by the use of abstract convexity.(Adilov & Tinaztepe, 2009)

There are some general schemes and approaches to abstract convexity as well as its applications. Here in this paper we show that these schemes can be implemented and can serve in the study of different problems in such areas as optimization, non smooth analysis, inequalities, theory of quasiconvex and monotonic functions. Also we consider some abstract convexities that have been used in the literature to generalize some results on the existence of continuous selections and fixed points to correspondences. In this framework we focus on abstract convexity structure called mc-spaces which is based on the idea of substituting the segment that joins any pair of points by a set that plays their role and study the relationship between it and L-convexity. (Ben-El-Mechaiekh et al., 1998). which is equivalent to it, simplicial convexity(Bielawski, 1987).. c-spaces,(Horvath, 1991, 1993)., B'- Simplicial convexity(Ben-El-Mechaiekh et al., 1998) and the convexity induced by an order.(Horvath & LLinares, 1996).

The paper formulation is as follows: In the next section we define abstract convexity, Minkowski Duality and ISSI functions. In section 3, we describe in brief about sub differential and abstract sub differential. Section 4 describes different kinds of abstract convexities- order convexity, c-Spaces, L- Space and mc- Space. In section 5, Simplicial convexity and B'- Simplicial Convexity is given. In section 6, a brief description of K-convex structure is given. In section 7, relationships between the abstract convexities are explained. At last some applications of these abstract convexities are given.

ABSTRACT CONVEXITIES, MINKOWSKY DUALITY, ISSI FUNCTION

In this section, we present some particular abstract convexities that appear in the literature, in relation to the problem of the existence of continuous selections and fixed points to correspondences. The general notion of abstract convexity structure is as follows:

Definition 1. Let H be a set of functions defined on a set X. Then the function f defined as follows on X is called abstract convex with respect to H or H-convex.

$$f(x) = sup\{h(x): h is an affine function, h \leq f\}$$

Definition 2. A set $U \subset H$ is called $(H, X) - convex$ if $\exists$ a function $f: X \to \overline{R}$ such that $U = s(f, H)$. Here the set of all $(H, X) - convex$ sets is denoted by $S(H, X)$.

Definition 3. A family C of subsets of a set X is an abstract convexity structure for X if $\emptyset$ and X belong to C and C is closed under arbitrary intersections.

The elements of C are called C-convex or simply abstract convex subsets of X and the pair (X, C) is called a convex space. Moreover the abstract convexity notion allows us to define the notion of the convex hull operator, which is similar to that of the closure operator in topology.

Definition 4. If X is a set with an abstract convexity C and A is a subset of X, then the hull operator generated by a convexity structure C which will denote by C_C and call C-convex hull , is defined for any subset $A \subseteq X$ by $C_C(A) = \cap \{B \in C: A \subseteq B\}$.

This operator enjoys certain properties that are identical to those of usual convexity: for example, $C_C(A)$ is the smallest C-convex set that contains A.

There are two definitions of convexity for closed sets. One of them is the inner definition based on the notion of convex combination and the other outer definition is based on the separation property. Abstract convexity leads to a generalization of the outer definition.

Symmetrically, we can also define H-concave functions.

Definition 5. A function $f: X \to \overline{R}$ is said to be abstract concave or H-concave if $\exists$ a set $U \subset H$ such that $f(x) = inf\{h(x): h \in U\} \forall\ x \in X$.

Definition 6. Let $P(H, X)$ be the set of all H-convex functions. Then a $P(H, X)$ $-$concave function $f: X \to \overline{R}$ is called inf- H –convex.

Definition 7. The mapping $\phi: P(H, X) \to S(P, H)$ defined by $\phi(p) = s(p, H)$ is called the Minkowski Duality. It is an isomorphism between the ordered sets $P(H, X)$ and $S(H, X)$.

Definition 8. The ISSI functions are increasing non-convex functions whose level sets are star shaped with respect to infinity. These are abstract convex with respect to min-type functions.

SUBDIFFERENTIAL AND ABSTRACT SUBDIFFERENTIAL

One of the main notions which play a key role in various applications, is the sub differential. There are two equivalent definitions of the subdifferential for a convex function. The first of them is based on the global behavior of the function whereas the second definition has a local nature and is connected with a local approximation of the function. For a differentiable convex functions these two definitions represent support and tangent sides of the gradient respectively.

Definition 9. A linear function l is called a member of the subdifferential that is a sub gradient of the function f at a point y if the affine function $h(x) = l(x) - \big(l(y) - f(y)\big)$ is a support function with respect to f, that is $h(x) \leq f(x) \forall\ x$.

Definition 10. The sub differential is a closed convex set of linear functions such that the directional derivative is the upper envelope of this set.

The various generalizations of the second definition have led to the development of non smooth analysis. The natural field for generalizations of the fist definition is abstract convexity. Now we give the definition of the abstract sub differential for abstract convex functions.

Definition 11. Let H be a set of real valued elementary functions defined on a set X. A function $h \in H$ is called the abstract subgradient of an H-convex function f at a point y if $f(x) \geq h(x) - (h(y) - f(y)) \forall x$. The set $\partial_H f(y)$ of all abstract subgradients of f at y is referred to as the abstract subdifferential or H- subdifferential of the function f at the point y.

ORDER CONVEXITY, c-SPACES , L- SPACES , mc- SPACES

ORDER CONVEXITY

Definition 12. If $(X, \leq)$is a partially ordered set and $\forall x, y \in X$, the closed interval is denoted by $[x, y] = \{z \in X: x \leq z \leq y\}$, so that it is possible to define an abstract convexity structure on X, called order convexity by considering the abstract convex sets like $Z \subseteq X$, such that $\forall x, y \in Z, [x, y] \subseteq Z$.(Horvath &LLinares, 1996).
Moreover, if $(X, \leq)$ is a (sup) semilattice and the supremum of $(x, y)$is denoted by $x \vee y$, then it is possible to consider the abstract convex sets like $Z \subseteq X$, such that $\forall x, y \in Z, [x, x \vee y] \cup [y, x \vee y] \subseteq Z$.

c-SPACES

We can consider some abstract convexities on a set X by associating to any finite family of points in X, a subset of X. This subset is in some sense, the generalized convex hull of these points. This is the case of notion of$c$-Space which associates in infinitely connected set C^{∞}, that satisfies some monotonicity conditions to any finite subset of X.(Horvath, 1993). The notion of c-Space is as follows:

Definition 13. If X is a topological space and $\langle X \rangle$ denotes the family of non-empty finite subsets of X, then a c-structure on X is given by a non-empty set valued map $\Gamma: \langle X \rangle \to X$ that satisfies:

1. $\forall A \in \langle X \rangle, \Gamma(A)$is non-empty and infinitely connected.
2. $\forall A, B \in \langle X \rangle, A \subset B \Rightarrow \Gamma(A) \subseteq \Gamma(B)$.

The pair (X, Γ) is called c-Space, and a subset $Z \subseteq X$ is called an H –set iff, it is satisfied $\forall A \in \langle Z \rangle, \Gamma(A) \subseteq Z$.
In the context of topological vector spaces, this definition includes as a particular case the notion of usual convexity. The family of H –sets define an abstract convexity on X.

L- SPACES

The notion of L- Space is a different abstract convexity that appears in the context of existence of continuous selections and fixed points to correspondences and which generalizes the B'- Simplicial convexity as well as the notion of c-Spaces.

Definition 14. An L- structure on X is given by a nonempty set- valued map $\Gamma: \langle X \rangle \to X$, such that for every $A \in \langle X \rangle$, say $A = \{a_0, a_1, \ldots, a_n\}$, $\exists$ a continuous function $f^A: \Delta_n \to \Gamma(A)$such that $\forall J \subset \{0,1, \ldots n\}, f^A(\Delta_J) \subseteq \Gamma(\{a_i: i \in J\})$.
The pair (X, Γ), is then called L- Space and a subset Z of X, is called an L- convex set if $\forall A \in \langle Z \rangle$, then $\Gamma(A) \subset Z$.
Clearly the family of L- convex sets define an abstract convexity structure on X.

mc- SPACES

The notion of mc- Space is a generalization of K-convex continuous structures, which is obtained by relaxing the continuity condition on function K. Now the ideal is to associate, for any finite set of points, a family of functions requiring their composition to be a continuous function. The image of this composition generates a set associated with the finite set of points, in a similar way to the case of c-Spaces or simplicial convexity. However, in contrast to these cases, no monotonicity condition on the associated sets is now required.

Definition 15. A topological space X is an mc- Space if for every $A \in \langle X \rangle$, say $A = \{a_0, a_1, \ldots, a_n\}$, $\exists$ a family of elements $\{b_0, b_1, \ldots, b_n\} \subset X$ and a family of functions $P_i^A: X \times [0,1] \to X$, such that for $i = 0,1, \ldots n$, $P_i^A(x, 0) = x$, $P_i^A(x, 1) = b_i$, $\forall x \in X$ and function $G_A: [0,1]^n \to X$ given by $G_A(t_0, t_1, \ldots, t_{n-1}) = P_0^A(\ldots(P_{n-1}^A(b_n, 1), t_{n-1}), \ldots, t_0)$, is a continuous function.
The notion of mc- spaces range over a wide field of possibilities, since it can appear in completely different contexts. The mc- spaces are also extensions of K-convex continuous spaces. If an mc- structure is given, it is possible to define an abstract convexity, by considering the family of sets that are stable under function G_A.

SIMPLICIAL CONVEXITY ANDB'- SIMPLICIAL CONVEXITY

(i) SIMPLICIAL CONVEXITY

A different way of introducing an abstract convexity structure from a family of continuous functions is by associating a continuous function defined on the standard simplex that satisfies some conditions to any finite subset of X.(Bielawski, 1987).

Definition 16. If X is a topological space and Δ_k, the k-dimensional simplex, X has a simplicial convexity if for each $n \in N$ and for each $(x_1, x_2, \dots, x_n) \in X^n$, $\exists$a continuous function $\Phi[x_1, x_2, \dots, x_n]: \Delta_{n-1} \rightarrow X$ that satisfies

1. $\forall\, x \in X$, $\Phi[x](1) = x$
2. $\forall\, n \geq 2, \forall (x_1, x_2, \dots, x_n) \in X^n, \forall (t_1, t_2, \dots, t_n) \in \Delta_{n-1}$, if $t_i = 0$, then

$$\Phi[x_1, x_2, \dots, x_n](t_1, t_2, \dots, t_n) = \Phi[x_{-i}](t_{-i})$$

Where x_{-i} denotes that x_i is omitted in $(x_1, x_2, \dots, x_n)$.
Moreover, a subset Z of X is called a simplicial convex set iff, $\forall\, n \in N$ and $\forall (a_1, a_2, \dots, a_n) \in Z^n$ it is satisfied that ,$\forall\, u \in \Delta_{n-1}, \Phi[a_1, a_2, \dots, a_n](u) \in Z$.
The simplicial convex sets are stable under arbitrary intersections. They therefore define an abstract convexity structure.

(ii) B'- SIMPLICIAL CONVEXITY

The notion of B'- Simplicial convexity is an obvious generalization of the notion of Simplicial convexity, since we weaken the conditions required of the continuous function that defines the B'- Simplicial convexity. Moreover, the notion of B'- Simplicial convexity allows us to connect the notion of c-Spaces with that of simplicial convexity as well as with other notions of abstract convexities we will introduce later.
Definition 17. A topological space X has a B'- Simplicial convexity if for each $n \in N$ and for each $(x_1, x_2, \dots, x_n) \in X^n$, $\exists$ a continuous function $\Phi[x_1, x_2, \dots, x_n]: \Delta_{n-1} \rightarrow X$ satisfying that $\forall\, n \geq 2, \forall (x_1, x_2, \dots, x_n) \in X^n, \forall (t_1, t_2, \dots, t_n) \in \Delta_{n-1}$, if $t_i = 0$, then

$$\Phi[x_1, x_2, \dots, x_n](t_1, t_2, \dots, t_n) = \Phi[x_{-i}](t_{-i})$$

In this context a subset Z of X is called a B'- Simplicial convex set iff $\forall\, n \in N$ and $\forall (a_1, a_2, \dots, a_n) \in Z^n$ it is satisfied that ,$\forall\, u \in \Delta_{n-1}, \Phi[a_1, a_2, \dots, a_n](u) \in Z$.
It is obvious that the family of B'- Simplicial convex sets is an abstract convexity, and the convex hull induced by this convexity is a sub simplicial hull.(Wieczorek, 1991). Moreover, the abstract convex sets obtained from a sub simplicial hull are B'- Simplicial convex sets.

K-CONVEX STRUCTURE

The K-convex structure is based on the idea of considering functions that join pairs of points. Here the segments used in usual convexity are substituted by an alternative path, previously fixed on K.
Definition 18. A K-convex structure on the set X is given by a mapping $K: X \times X \times [0,1] \rightarrow X$. Furthermore (X, K) will be called a K-convex space and function K a K-convex function.
If (X, K) is a K-convex space, it is possible for any pair of points $x, y \in X$ to associate themselves with a subset given by $K(x, y, [0,1]) = \cup \{K(x, y, t): t \in [0,1]\}$(Prenowitz&Jantosciak, 1979) or with interval spaces (Stacho, 1980).
Moreover we can define an abstract convexity on X by considering a family, C, of subsets of X as follows:

$$Z \in C \Leftrightarrow \forall\, x, y \in Z \;\; K(x, y, [0,1]) \subseteq Z$$

The elements of C will be called K-convex sets and the K-convex hull operator associated to this family C of K-convex sets will be denoted by C_K.
A different case of K-convexity is that of the equiconnected spaces, introduced by Dugundji and Himmelberg which are a particular case of K-convex continuous spaces.(Himmelberg Ch. J. (1965) Dugundji,1965;Himmelberg, 1965)
Definition 19. A metric space X is equiconnectediff$\exists$ a continuous function $K: X \times X \times [0,1] \rightarrow X$ such that $\forall\, x, y \in X, K(x, y, 0) = x, K(x, y, 1) = y, K(x, x, t) = x$ for any $t \in [0,1]$.

RELATIONS BETWEENABSTRACT CONVEXITIES

In this section we give the proof of some theorems showing the relationship between some of the abstract convexity notions defined above.

RELATION BETWEENK-CONVEX CONTINUOUS SPACE ANDc-SPACE

Theorem 1. If (X, K) is a K-convex continuous space, then $\exists$ a nonempty set-valued map $\Gamma: \langle X \rangle \rightarrow X$ such that $(X, \Gamma)$is ac-Space and K-convex sets are H-sets.
Proof: If (X, K) is a K-convex continuous space, then we can define the mapping $\Gamma: \langle X \rangle \rightarrow X$, by means of the K-convex hull, that is $\Gamma(A) = C_K(A)$. Then by applying preposition 1.1 (LLinares,1995), we know that $\Gamma(A)$ is contractible. Moreover it is easy to prove that $\forall A, B \in \langle X \rangle$, if $A \subset B$, then $\Gamma(A) \subset \Gamma(B)$, So $(X, \Gamma)$is ac-Space.

Finally, to show that K-convex sets are H-sets, assume that $\exists$ a convex set Zsuch that $A\in\langle Z\rangle$and $\Gamma(A) = C_K(A)$ is not properly contained in Z. Then we have that $A\subset C_K(A)$, $A\subset Z$ and that both of them are K-convex sets, so $A\subset Z \cap C_K(A)$ is not properly contained in $C_K(A)$, which is in contradiction with the fact that $C_K(A)$is the smallest K-convex set containing A.

RELATIONSHIP BETWEENK-CONVEX CONTINUOUS STRUCTURES AND SIMPLICIAL CONVEXITIES

Theorem 2. If (X, K) is a topological space with a K-convex continuous structure, then it is possible to define a simplicial convexity on X such that K-convex sets are simplicial convex sets.

Proof: For any $n\in N$ and for any $(a_1, a_2, \dots, a_n)\in X^n$, we define the family of functions $\Phi[a_1, a_2, \dots, a_n]$ as follows,

If $n = 1$, Φ[a]=$K(a, a, 1)$ and for $n \geq 2$,

$$\Phi[a_1, a_2, \dots, a_n](t_1, t_2, \dots, t_n) = K(\dots K(K(a_n, a_{n-1}, t_{n-1}), a_{n-2}, t_{n-2}) \dots), a_1, t_1$$

It is easy to show that this family of functions defines a simplicial convexity on X that coincides with the one that is obtained from K.

RELATIONSHIP BETWEEN ORDER CONVEXITY STRUCTURE AND SIMPLICIAL CONVEXITY

Theorem 3. If $(X, \preceq)$ is a topological semilattice with path-connected intervals, then $\exists$ a Simplicial Convexity on X such that order convex sets are simplicial convex sets.

Proof: If $(X, \preceq)$ is a topological semilattice with path-connected intervals, then we can define a non-empty set valued map$\Gamma: \langle X\rangle \to X$, given by $\Gamma(A) = \cup_{a\in A}[a, \sup A]$. By applying lemma 2.1 (Horvath & LLinares, 1996). we know that for any $n\in N$, any continuous function $g: \partial\Delta_n \to \Gamma(A)$can be extended to a continuous function $f: \Delta_n \to \Gamma(A)$, so $\Gamma(A)$is C^∞. Therefore, if we define $hull\{A\} = \Gamma(A)$, the family of order convex sets is an abstract convexity such that $hull\{A\}$ is $C^\infty$and by preposition 1.5 (Bielawski, 1987), we obtain the conclusion.

RELATIONSHIP BETWEEN c-SPACES AND SIMPLICIAL CONVEXITYAND B'- SIMPLICIAL CONVEXITY

Theorem 4. If $(X, \Gamma)$is a c-Space such that $\forall\, x\in X, x\in\Gamma(\{x\})$, then X has a Simplicial Convexity such that H −sets are Simplicial Convex Sets.

Theorem 5. If $(X, \Gamma)$is a c-Space, then it is possible to define a B'- Simplicial Convexity such that H −sets are B'-Simplicial Convex Sets.

Theorem 6. If X is a topological space with a Simplicial Convexity, then this Simplicial Convexity defines a B'-Simplicial Convexity such that Simplicial Convex Sets are B'- Simplicial Convex Sets.

Proof: The Proofs of theorem 4, 5, 6. are obvious and can be immediately obtained from the definitions given above.

APPLICATIONS

Abstract convexity is a very convenient tool for studying many problems in different areas of mathematics. We begin with applications to optimization. Abstract convexity is very useful both in the theoretical study of optimization problems and in the development of numerical methods. In particular, abstract convexity sometimes allows us to obtain a conceptual trivialization of a problem under consideration. This means that we can obtain a conceptual trivialization of a problem by means of a theory if it is quite easy to obtain a clear understanding of the problem in the framework of the theory. Though there are technical obstacles and we need time to solve the problem but ultimately problem is solved.

Two examples of such trivialization by means of abstract convexity are Lagrange Multipliers Theory and Solvability Theorems. Approach to the Lagrange Multipliers Theory is based on the notion of the abstract sub differential.(Pallaschke & Rolewicz, 1997)**.** The most convenient tool which allows us to conceptually trivialize the study of solvability theorems is Minkowski Duality. Some important applications are as follows:

GENERAL SOLVABILITY THEOREM

We shall consider Conic set H of functions defined on a set X. For $V \subset H$, the conic hull cone V of V is defined by

$$\text{cone } V = \bigcup_{\lambda>0} \lambda V$$

We shall use the (H, X)-convex hull $co_H U$ of the set $U \subset H$.

Now we prove a theorem which allows us to obtain a clear understanding of solvability theorems in the framework of abstract convexity. It follows from this theorem that the main difficulties related to solvability theorems are in describing support sets and (H, X)-convex hulls where H is a given conic set of functions.

Theorem 1. Let H be a conic set of real valued functions defined on X and let I be any arbitrary index set. Let f and , for each $i \in I$, g_i be H-convex functions defined on X. Then the following statements are equivalent:

a) $(\forall i \in I) g_i(x) \leq 0 \Rightarrow f(x) \leq 0$

b) $s(f, H) \subset co_H \bigcup_{i \in I} cone\, s(g_i, H)$

Proof: Let $g(x) = sup_i g_i(x)$. Clearly g is an H-convex function. Let $S_g = \{x: g(x) \leq 0\}$, $S_f = \{x: f(x) \leq 0\}$. Consider the indicator functionδ of the set S_g:

$$\delta(x) = \begin{cases} +\infty & x \notin S_g \\ 0 & x \in S_g \end{cases}$$

Thus we have

a) $\Leftrightarrow S_g \subset S_f \Leftrightarrow f \leq \delta \Leftrightarrow s(f, H) \subset s(\delta, H)$

Since $\delta = \sup_{\lambda>0} \lambda g = \sup_{\lambda>0 i \in I} \lambda g_i$

and Minkowski duality is an isomorphism between complete lattices $P(H, X)$ and $S(H, X)$,it follows that

$$s(\delta, H) = co_H \bigcup_{\lambda>0 i \in I} s(g_i, H) = co_H \bigcup_{i \in I} cone\, s(g_i, H).$$

Hence the theorem is proved.

NONSMOOTH ANALYSIS

For a local approximation of a nonsmooth function various constructions are used which lead to a non-linear approximation of the first order and to a non quadratic approximation of the second order of a given function. Very often an approximation is accomplished by various kinds of generalized derivatives of the first order which are positively homogeneous functions of the first degree. In order to apply these derivatives one should express them in terms of linear functions.

Methods of abstract convexity (namely Φ_2-convexity and convexity with respect to the set $S(n)$of all symmetric operators) allows one to obtain deep results in the study of the second order approximation of nonsmooth functions both in finite dimensional and infinite dimensional cases. In particular the inf- convolution in the abstract convex setting is a very useful tool in the study of such problems.

QUASICONVEX FUNCTIONS

Abstract convexity is a very useful tool in the study of quasiconvex functions. Various schemes for the application of abstract convexity to quasiconvexity were suggested.(Legaz, 1988; Penot&Volle, 1998,1990; Singer, 1997; Volle, 1985). In particular, the following topics have been discussed:

- Various kinds of supremal generators of the set of all lower semi-continuous quasiconvex functions or the set of all evenly quasiconvex functions and also supremal generators of various subsets of these sets.
- Various kinds of conjugations in quasiconvex analysis; many of them can be considered from the abstract convex analysis point of view, by applying the level sets conjugation.
- Various kinds of sub differentials; many of them can also be considered from the abstract convexity point of view.

APPLICATIONS TO ECONOMICS

Various classes of monotonic functions as well as quasiconvex functions are used in mathematical economics. To illustrate the motivation for various classes of abstract convex functions from mathematical economics, we consider the commodity space R_+^n. Some important functions which are used in economics are production functions and utility functions.

A production function $f: R_+^n \to R_+$describes an output $f(x)$ of the economical system as a function of its input $x \in R_+^n$. As a rule increasing production functions are considered. One of the main characteristics of production functions is the so-called returns to scale. A function f is said to have the constant returns to scale (respectively , decreasing returns to scale , increasing returns to scale) if $f(\alpha x) = \alpha f(x) \forall \alpha > 1$(respectively $f(\alpha x) < \alpha f(x) \forall \alpha > 1, f(\alpha x) > \alpha f(x) \forall \alpha > 1$). It is easy to check that $f$has constant returns to scale iff $f(\alpha x) = \alpha f(x) \forall \alpha > 0$ and $f$has decreasing returns to scale iff $(\alpha x) > \alpha f(x) \forall \alpha \in (0,1]$. Thus an increasing production function with constant return to scale (respectively, decreasing returns to scale) is an IPH function (Increasing Positively Homogeneous Function). In the same manner, f has increasing returns to scale iff $f(\alpha x) < \alpha f(x) \forall \alpha \in (0,1]$.

Now we consider utility functions. Since we assume that all goods are essentially useful , it follows that a utility function increases. One of the main properties of a utility function u can be expressed in the following form: $u(\alpha x) \leq u f(x), \forall\, x \in R_+^n, \forall \alpha > 1$. This is the so- called Law of diminishing marginal utility. Thus a utility function is an ISSI function. Hence we should actually consider quasiconcave ISSI functions as utility functions.

CONCLUSION

This paper well establishes the relationship between abstract convexities and discusses applications in economics and general solvability theorem.

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